Bisimulations for Probabilistic and Quantum Processes

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Abstract

Bisimulation is a fundamental concept in the classical concurrency theory for comparing the behaviour of nondeterministic processes. It admits elegant characterisations from various perspectives such as fixed point theory, modal logics, game theory, coalgebras etc. In this paper, we review some key ideas used in the formulations and characterisations of reasonable notions of bisimulations for both probabilistic and quantum processes. To some extent the transition from probabilistic to quantum concurrency theory is smooth and natural. However, new ideas need also to be introduced. We have not yet reached the stage of formally verifying quantum communication protocols and quantum algorithms using bisimulations implemented by automatic tools. We discuss some recent efforts in this direction.

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1 Introduction

Bisimulation [39, 37] is a fundamental concept in the classical concurrency theory as it admits beautiful characterisations in terms of fixed points, modal logics, co-algebras, pseudometrics, games, decision algorithms, etc. Its generalisation in the probabilistic setting is initiated by Larsen and Skou in [36] and has subsequently been widely investigated in probabilistic concurrency theory. One of the main contributions of [36] is the introduction of a lifting operation that converts a relation between states to a relation between distributions over states. Later on, the lifting operation is shown to be closely related to some prominent concepts in mathematics such as the Kantorovich metric [33, 45] and the maximum network flow problem [1]; the latter is crucial for designing algorithms to check if two states are bisimilar.

The probabilistic bisimulation nicely defined in [36] has natural characterisations by probabilistic extensions of Hennessy-Milner logic [28]; see e.g. [36, 14, 15, 40, 10, 30, 25, 12, 4]. Most characterisations employ some modalities indexed with numbers. A typical modal formula, dated back to [36], is $\langle a \rangle_p$, where p is a probability value. A state s satisfies this

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The process constructs we give here are quite similar to those in classical CCS, and they also have similar intuitive meanings: $\mathbf{nil}$ stands for a process which does not perform any action; $c?x$ and $c!e$ are respectively classical input and classical output, while $\mathtt{c}?q$ and $\mathtt{c}!q$ are their quantum counterparts. $E[\widetilde{q}]$ denotes the action of performing the super-operator E on the qubits $\widetilde{q}$ while $M[\widetilde{q};x]$ measures the qubits $\widetilde{q}$ according to M and the measurement outcome is substituted for the classical variable x. The binary sum + models nondeterministic choice: $U + V$ behaves like either U or V depending on the choice of the environment. denotes the usual parallel composition. Finally, $\mathbf{if}\ b\ \mathbf{then}\ U$ is the standard conditional choice where U can be executed only if b is $\mathtt{tt}$.

We now turn to the operational semantics of qCCS. For each quantum variable q *qVar*, we assume a 2-dimensional Hilbert space H_q to be the state space of the q-system. For any S *qVar*, we denote $H_S = \bigotimes_{q\ S} H_q$. In particular, $H = H_{qVar}$ is the state space of the whole environment consisting of all the quantum variables. Note that H is a countably-infinite dimensional Hilbert space.

Suppose P is a closed quantum process. A pair of the form P, ρ is called a *configuration*, where ρ $D(H)$ is a density operator on H (As H is infinite dimensional, ρ should be understood as a density operator on some finite dimensional subspace of H which contains $H_{qv(P)}$). The set of configurations is denoted by Con, and ranged over by $C, D, \cdots$. Let

$$Act = \{\tau\} \quad \{c?v, c!v \mid c \quad \textit{cChan}, v \quad \mathsf{Real}\} \quad \{\mathtt{c}?r, \mathtt{c}!r \mid \mathtt{c} \quad \textit{qChan}, r \quad \textit{qVar}\}.$$

Let $D(Con)$, ranged over by $\Delta, \Theta, \cdots$, be the set of all finite-supported probabilistic distributions over Con. Then the operational semantics of qCCS can be given by the pLTS $Con, Act, \longrightarrow$, where $\longrightarrow$ $Con \times Act \times D(Con)$ is the smallest relation satisfying some inference rules. Here we select two rules related to super-operator application and quantum measurements; the others can be found in [5].

$$\textit{(Oper)}\quad E[\tilde{q}].P, \rho \xrightarrow{\tau} P, E_{\tilde{q}}(\rho) \qquad\qquad \textit{(Meas)}\quad \frac{M = \sum_{i\ I} \lambda_i E^i \qquad p_i = tr(E^i_{\tilde{q}}\rho)}{M[\widetilde{q};x].P, \rho \xrightarrow{\tau} \sum_{i\ I} p_i\ P[\lambda_i/x], E^i_{\widetilde{q}}\rho E^i_{\widetilde{q}}/p_i}$$

In rule *(Meas)*, $E^i_{\tilde{q}}$ denotes the operator E^i acting on the quantum systems $\tilde{q}$ and $tr(E^i_{\tilde{q}}\rho)$ stands for the trace of $E^i_{\tilde{q}}\rho$. This rule tells us that a measurement on the quantum system $\tilde{q}$ entails a probabilistic transition; each candidate configuration $P[\lambda_i/x], E^i_{\tilde{q}}\rho E^i_{\tilde{q}}/p_i$ occurs with probability $tr(E^i_{\tilde{q}}\rho)$.

Let $C = P, \rho$. We use the notation $qv(C) := qv(P)$ for free quantum variables and $\mathsf{env}(C) := \mathsf{tr}_{qv(P)}(\rho)$ for partial traces. Let $\Delta = \sum_{i\ I} p_i \cdot \overline{P_i, \rho_i}$. We write $E(\Delta)$ for the distribution $\sum_{i\ I} p_i \cdot \overline{P_i, E(\rho_i)}$. In addition, we let $qv(\Delta) := \bigcup_{i\ I} qv(P_i)$ and $\mathsf{env}(\Delta) := \sum_{i\ I} p_i \cdot \mathsf{tr}_{qv(P_i)}(\rho_i)$.

▶ **Definition 9.** A symmetric relation R $Con \times Con$ is called an open bisimulation if for any C, D Con, $C\ R\ D$ implies that

(1) $qv(C) = qv(D)$, and $\mathsf{env}(C) = \mathsf{env}(D)$,

(2) for any trace-preserving super-operator E acting on $H_{\overline{qv(C)}}$ (Again, E should be understood as a super-operator on some finite dimensional subspace of $H_{\overline{qv(C)}}$), whenever $E(C) \xrightarrow{\alpha} \Delta$, there exists Θ such that $E(D) \xrightarrow{\alpha} \Theta$ and $\Delta\ R^\dagger\ \Theta$.

Two quantum configurations C and D are open bisimilar, denoted by $C\ _o\ D$, if there exists an open bisimulation R such that $C\ R\ D$.

Here we are using exactly the same lifting operation as that in the probabilistic case (cf. Definition 2). The above definition is inspired by the work of Sangiorgi [42], where a

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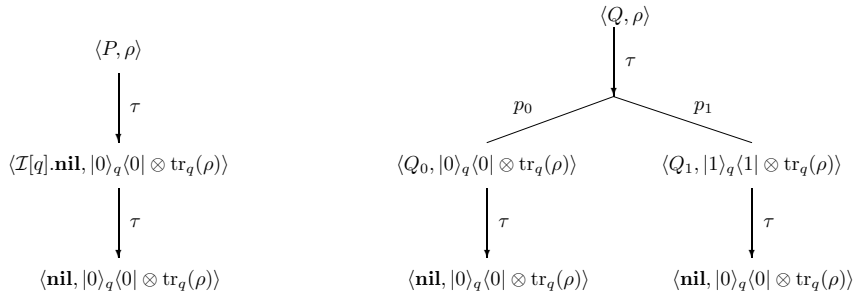

Figure 2 pLTSs for the two ways of setting a quantum system to $|0\rangle$

notion of bisimulation is defined for the π-calculus [38, 42] by treating name instantiation in an "open" style (name instantiation happens before any transition). Here we deal with super-operator application in an "open" style, but the instantiation of variables can be in an "early" style (variables are instantiated when input actions are performed). For example, the operational semantics given in [5] is essentially an early semantics.

To illustrate the operational semantics and open bisimulation presented in this section, we give a simple example.

▶ **Example 10.** This example shows two alternative ways of setting a quantum system to the pure state $|0\rangle$. Let $P \stackrel{def}{=} Set^0[q].I[q].\mathbf{nil}$ and

$$Q \stackrel{def}{=} M_{0,1}[q; x].(\mathbf{if}\ x = 0\ \mathbf{then}\ I[q].\mathbf{nil}\ +\ \mathbf{if}\ x = 1\ \mathbf{then}\ X[q].\mathbf{nil}),$$

where $Set^0 = \{|0\rangle\ |0\rangle, |0\rangle\ |1\rangle\}$, $M_{0,1}$ is the 1-qubit measurement according to the computational basis $\{|0\rangle, |1\rangle\}$, I is the identity super-operator, and X is the Pauli-X super-operator. For any ρ $D(H)$, the pLTSs rooted by P, ρ and Q, ρ respectively are depicted in Figure 2 where

$$Q_0 \stackrel{def}{=} \mathbf{if}\ 0 = 0\ \mathbf{then}\ I[q].\mathbf{nil}\ +\ \mathbf{if}\ 0 = 1\ \mathbf{then}\ X[q].\mathbf{nil},$$

$$Q_1 \stackrel{def}{=} \mathbf{if}\ 1 = 0\ \mathbf{then}\ I[q].\mathbf{nil}\ +\ \mathbf{if}\ 1 = 1\ \mathbf{then}\ X[q].\mathbf{nil},$$

and $p_i = \mathrm{tr}(|i\rangle\ i|_q \cdot \rho)$. Note that both P and Q are free of quantum input. We can show $P\ _o\ Q$ easily by verifying that the relation $R\ R^{-1}$, where

$$R = \{(P, \rho\ ,\ Q, \rho\ ), (\ I[q].\mathbf{nil}, \rho_0\ ,\ Q_0, \rho_0\ ),$$
$$(\ I[q].\mathbf{nil}, \rho_0\ ,\ Q_1, \rho_1\ ), (\ \mathbf{nil}, \rho_0\ ,\ \mathbf{nil}, \rho_0\ ) : \rho\ D(H)\}$$

and $\rho_i = |i\rangle\ i|_q\ \mathrm{tr}_q\rho$, is an open bisimulation.

3.2 A Useful Proof Technique

In Definition 9 super-operator application and transitions are considered at the same time. In fact, we can separate the two issues and approach the concept of open bisimulation in an incremental way, which turns out to be very useful when proving that two configurations are bisimilar.

▶ **Definition 11.** A relation R $Con \times Con$ is closed under super-operator application if $C\ R\ D$ implies $E(C)\ R\ E(D)$ for any trace-preserving super-operator E acting on $H_{\overline{qv(C)}}$.

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▶ **Definition 12.** A relation $R \subseteq Con \times Con$ is a *ground simulation* if $C\ R\ D$ implies that $qv(C) = qv(D)$, $\mathrm{env}(C) = \mathrm{env}(D)$, and

- whenever $C \xrightarrow{\alpha} \Delta$, there is some distribution Θ with $D \xrightarrow{\alpha} \Theta$ and $\Delta\ R^{\dagger}\ \Theta$.

A relation R is a *ground bisimulation* if both R and R^{-1} are ground simulations.

The following property is shown in [5].

▶ Proposition 13. $\sim_o$ is the largest ground bisimulation that is closed under all super-operator applications.

Proposition 13 provides us with a useful proof technique: in order to show that two configurations C and D are open bisimilar, it suffices to exhibit a binary relation including the pair (C, D), and then to check that the relation is a ground bisimulation and is closed under all super-operator application. This is analogous to a proof technique of open bisimulation for the π-calculus [42], where name instantiation is playing the same role as super-operator application here.

3.3 Distribution-Based Quantum Bisimulation

The distribution-based bisimulation defined in Section 2.4 can also be extended to the quantum setting.

▶ **Definition 14.** A relation $R \subseteq \mathcal{D}(Con) \times \mathcal{D}(Con)$ is a *distribution-based ground simulation* if $\Delta\ R\ \Theta$ implies that $qv(\Delta) = qv(\Theta)$, $\mathrm{env}(\Delta) = \mathrm{env}(\Theta)$, and

- whenever $\Delta \xrightarrow{\alpha} \Delta'$, there is some subdistribution Θ' with $\Theta \xrightarrow{\alpha} \Theta'$ and $\Delta'\ R\ \Theta'$.

A relation R is a *distribution-based ground bisimulation* if both R and R^{-1} are distribution-based ground simulations.

A relation R is a distribution-based bisimulation if it is a distribution-based ground bisimulation, and is closed under super-operator applications.

Note that the distribution-based bisimulation given in Definition 14 is slightly coarser than that considered in [22], for the same reason as the comparison of the corresponding probabilistic bisimulations [16].

In quantum mechanics, a fundamental principle is the no-cloning theorem of quantum resources. From a type-theoretic point of view, quantum resources are linear and can be described by linear types in quantum programming languages. How to define appropriate program equivalences for this kind of languages is an interesting problem. In [8] a linear contextual equivalence is introduced to compare the behaviour of quantum programs. Two notions of bisimilarity, a state-based and a distribution-based are introduced as proof techniques for reasoning about higher-order quantum programs. Both notions of bisimilarity are sound with respect to the linear contextual equivalence, but only the distribution-based one turns out to be complete.

3.4 Symbolic Bisimulations

The quantum bisimulations introduced so far, either state-based or distribution-based, are generalised from the corresponding probabilistic bisimulations naturally and smoothly. A major problem with them is that they all resort to the instantiation of quantum variables by quantum states. As a result, to check whether or not two processes are bisimilar, we have to accompany them with arbitrarily chosen quantum states, and check if the resultant configurations are bisimilar. Note that all quantum states constitute a continuum. Therefore,

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it seems that the verification of quantum bisimulations is infeasible from an algorithmic point of view.

Recall that for classical process algebras, Hennessy and Lin [27] introduced a notion of symbolic bisimulation to deal with possibly infinite classical data sets. As a quantum extension of value-passing CCS, the quantum process algebra qCCS has both (possibly infinite) classical data domain and (doomed-to-be infinite) quantum data domain. To overcome the additional difficulty caused by the infinity of all quantum states, we can make use of super-operator valued distributions, which allow us to fold the operational semantics of qCCS into a symbolic version and thus provide us with a notion of symbolic bisimulation. To check the symbolic bisimilarity of two quantum processes, only a finite number of process-superoperator pairs need to be considered, without appealing to quantum states. This idea has been successful in developing an algorithm to check the state-based ground bisimulation for quantum processes [18]. It would be interesting to purse this line of research so as to develop algorithms of checking the symbolic versions of other quantum bisimulations.

4 Concluding Remarks

We have briefly reviewed a few ingredients for formulating reasonable notions of probabilistic and quantum bisimulations.

(1) The lifting operation is the key of defining state-based probabilistic and quantum bisimulations. It is mathematically interesting in itself because of the close connection with the Kantorovich metric and the maximum network flow problem.

(2) Distribution-based bisimulation is more relevant to quantum processes because it offers a coinductive proof technique for linear contextual equivalene, and linear resources are prominent in quantum computation.

(3) The symbolic approach is promising to yield feasible algorithms of checking quantum bisimulations.

There is a huge amount of literature on probabilistic bisimulations, and the current paper is by no means a complete survey. A more detailed account of probabilistic bisimulations is given in [4, Chapter3]. For quantum processes, a branching bisimulation is firstly proposed in [35]. However, it is not a congruence because it is not preserved by parallel composition. Quantum bisimulations that are congruence relations are given in [20, 18] and independently in [3]. Both of them are defined for concrete quantum transition systems, and are difficult to check with algorithms, which motivated the introduction of symbolic bisimulations for quantum processes [18].

In [34] a semi-automated tool is developed to verify security proofs based on a weak bisimulation similar to that given in Definition 9 for a finite fragment of qCCS. In that tool, security parameters and quantum states are represented as symbols, and some user-defined equations are used as rewriting rules for simplification. This differs from the symbolic semantics discussed in Section 3.4 as the latter is more in line with the idea investigated in [27] for value-passing CCS.

In the future, we believe that distribution-based symbolic bisimulations would be promising to be used in software tools in support of verifying quantum communication protocols. Some efforts are made in [22], which considers distribution-based bisimulations and the proofs are manual when reasoning about the behavioural equivalence of quantum processes. In order to deal with advanced protocols such as the quantum key distribution protocol BB84 [2], it would be helpful to have some tool support, for which symbolic semantics could play a role.

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References

1. C. Baier, B. Engelen, and M. E. Majster-Cederbaum. Deciding bisimilarity and similarity for probabilistic processes. *Journal of Computer and System Sciences*, 60(1):187–231, 2000.
2. C. Bennett and G. Brassard. Quantum cryptography: Public key distribution and coin tossing. In *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing*, pages 175–179, 1984.
3. T. Davidson. *Formal verification techniques using quantum process calculus*. PhD thesis, University of Warwick, 2011.
4. Y. Deng. *Semantics of Probabilistic Processes: An Operational Approach*. Springer, 2015.
5. Y. Deng and Y. Feng. Open bisimulation for quantum processes. In *Proceedings of the 7th IFIP International Conference on Theoretical Computer Science*, volume 7604 of *LNCS*, pages 119–133. Springer, 2012.
6. Y. Deng and Y. Feng. Bisimulations for probabilistic linear lambda calculi. In *Proceedings of the 11th IEEE International Symposium on Theoretical Aspects of Software Engineering*, pages 1–8. IEEE Computer Society, 2017.
7. Y. Deng and Y. Feng. Probabilistic bisimilarity as testing equivalence. *Information and Computation*, 257:58–64, 2017.
8. Y. Deng, Y. Feng, and U. D. Lago. On coinduction and quantum lambda calculi. In *Proceedings of the 26th International Conference on Concurrency Theory*, volume 42 of *LIPIcs*, pages 427–440. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015.
9. Y. Deng and M. Hennessy. On the semantics of Markov automata. *Information and Computation*, 222:139–168, 2013.
10. Y. Deng and R. van Glabbeek. Characterising probabilistic processes logically. In *Proceedings of the 17th International Conference on Logic for Programming, Artificial Intelligence and Reasoning*, volume 6397 of *LNCS*, pages 278–293. Springer, 2010.
11. Y. Deng, R. van Glabbeek, M. Hennessy, and C. Morgan. Testing finitary probabilistic processes (extended abstract). In *Proceedings of the 20th International Conference on Concurrency Theory*, volume 5710 of *LNCS*, pages 274–288. Springer, 2009.
12. Y. Deng and H. Wu. Modal characterisations of probabilistic and fuzzy bisimulations. In *Proceedings of the 16th International Conference on Formal Engineering Methods*, volume 8829 of *LNCS*, pages 123–138. Springer, 2014.
13. J. Desharnais. *LabelledMarkovProcesses*. PhD thesis, McGillUniversity, 1999.
14. J. Desharnais, A. Edalat, and P. Panangaden. Bisimulation for labelled Markov processes. *Information and Computation*, 179(2):163–193, 2002.
15. J. Desharnais, V. Gupta, R. Jagadeesan, and P. Panangaden. Approximating labelled Markov processes. *Information and Computation*, 184(1):160–200, 2003.
16. W. Du, Y. Deng, and D. Gebler. Behavioural pseudometrics for nondeterministic probabilistic systems. In *Proceedings of the the 2nd International Symposium on Dependable Software Engineering: Theories, Tools, and Applications*, volume 9984 of *LNCS*, pages 67–84. Springer, 2016.
17. C. Eisentraut, J. C. Godskesen, H. Hermanns, L. Song, and L. Zhang. Probabilistic bisimulation for realistic schedulers. In *Proceedings of the 20th International Symposium on Formal Methods*, volume 9109 of *LNCS*, pages 248–264. Springer, 2015.
18. Y. Feng, Y. Deng, and M. Ying. Symbolic bisimulation for quantum processes. *ACM Transactions on Computational Logic*, 15(2):1–32, 2014.
19. Y. Feng, R. Duan, Z. Ji, and M. Ying. Probabilistic bisimulations for quantum processes. *Information and Computation*, 205(11):1608–1639, 2007.
20. Y. Feng, R. Duan, and M. Ying. Bisimulations for quantum processes. In M. Sagiv, editor, *Proceedings of the 38th ACM Symposium on Principles of Programming Languages*, pages 523–534, Austin, Texas, USA, 2011.

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21 Y. Feng, R. Duan, and M. Ying. Bisimulation for Quantum Processes. *ACM Transactions on Programming Languages and Systems*, 34(4):1–43, Dec. 2012.

22 Y. Feng and M. Ying. Toward automatic verification of quantum cryptographic protocols. In *26th International Conference on Concurrency Theory*, volume 42 of *LIPIcs*, pages 441–455. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015.

23 Y. Feng and L. Zhang. When equivalence and bisimulation join forces in probabilistic automata. In *Proceedings of the 19th International Symposium on Formal Methods*, volume 8442 of *LNCS*, pages 247–262. Springer, 2014.

24 S. J. Gay and R. Nagarajan. Communicating quantum processes. In *Proceedings of the 32nd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, pages 145–157. ACM, 2005.

25 M. Hennessy. Exploring probabilistic bisimulations, part I. *Formal Aspects of Computing*, 24(4-6):749–768, 2012.

26 M. Hennessy and A. Ingólfsdóttir. A theory of communicating processes value-passing. *Information and Computation*, 107(2):202–236, 1993.

27 M. Hennessy and H. Lin. Symbolic bisimulations. *Theoretical Computer Science*, 138(2):353–389, 1995.

28 M. Hennessy and R. Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, 32(1):137–161, 1985.

29 H. Hermanns, J. Krcál, and J. Kretínský. Probabilistic bisimulation: Naturally on distributions. In *Proceedings of the 25th International Conference on Concurrency Theory*, volume 8704 of *LNCS*, pages 249–265. Springer, 2014.

30 H. Hermanns, A. Parma, R. Segala, B. Wachter, and L. Zhang. Probabilistic logical characterization. *Information and Computation*, 209(2):154–172, 2011.

31 C. Jones. *Probabilistic nondeterminism*. PhD thesis, University of Edinburgh, 1990.

32 P. Jorrand and M. Lalire. Toward a quantum process algebra. In *Proceedings of the First Conference on Computing Frontiers*, pages 111–119. ACM, 2004.

33 L. Kantorovich. On the transfer of masses (in Russian). *Doklady Akademii Nauk*, 37(2):227–229, 1942.

34 T. Kubota, Y. Kakutani, G. Kato, Y. Kawano, and H. Sakurada. Semi-automated verification of security proofs of quantum cryptographic protocols. *Journal of Symbolic Computation*, 73:192–220, 2016.

35 M. Lalire. Relations among quantum processes: bisimilarity and congruence. *Mathematical Structures in Computer Science*, 16(3):407–428, 2006.

36 K. G. Larsen and A. Skou. Bisimulation through probabilistic testing. *Information and Computation*, 94:1–28, 1991.

37 R. Milner. *Communication and Concurrency*. Prentice Hall, 1989.

38 R. Milner. *Communicating and mobile systems - the Pi-calculus*. Cambridge University Press, 1999.

39 D. Park. Concurrency and automata on infinite sequences. In *Proceedings of the 5th GI Conference*, volume 104 of *LNCS*, pages 167–183. Springer, 1981.

40 A. Parma and R. Segala. Logical characterizations of bisimulations for discrete probabilistic systems. In *Proceedings of the 10th International Conference on Foundations of Software Science and Computational Structures*, volume 4423 of *LNCS*, pages 287–301. Springer, 2007.

41 J. Sack and L. Zhang. A general framework for probabilistic characterizing formulae. In *Proceedings of the 13th International Conference on Verification, Model Checking, and Abstract Interpretation*, volume 7148 of *LNCS*, pages 396–411. Springer, 2012.

42 D. Sangiorgi. A theory of bisimulation for the pi-calculus. *Acta Informatica*, 33(1):69–97, 1996.

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43 R. Segala and N. Lynch. Probabilistic simulations for probabilistic processes. In *Proceedings of the 5th International Conference on Concurrency Theory*, volume 836 of *LNCS*, pages 481–496. Springer, 1994.

44 F. van Breugel, M. W. Mislove, J. Ouaknine, and J. Worrell. Domain theory, testing and simulation for labelled Markov processes. *Theoretical Computer Science*, 333(1-2):171–197, 2005.

45 F. van Breugel and J. Worrell. An algorithm for quantitative verification of probabilistic transition systems. In *Proceedings of the 12th International Conference on Concurrency Theory*, volume 2154 of *LNCS*, pages 336–350. Springer, 2001.

46 R. J. van Glabbeek, S. A. Smolka, B. Ste□en, and C. M. N. Tofts. Reactive, generative, and stratified models of probabilistic processes. In *Proceedings of the 5th Annual Symposium on Logic in Computer Science*, pages 130–141. IEEE Computer Society, 1990.

47 M. Ying, Y. Feng, R. Duan, and Z. Ji. An algebra of quantum processes. *ACM Transactions on Computational Logic*, 10(3):1–36, 2009.

48 L. Zhang, H. Hermanns, F. Eisenbrand, and D. N. Jansen. Flow faster: E□cient decision algorithms for probabilistic simulations. *Logical Methods in Computer Science*, 4(4):1–43, 2008.

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