Modal Characterisations of Probabilistic and Fuzzy Bisimulations

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Abstract. This paper aims to investigate bisimulation on fuzzy systems. For that purpose we revisit bisimulation in the model of reactive probabilistic processes with countable state spaces and obtain two findings: (1) bisimilarity coincides with simulation equivalence, which generalises a result on finite-state processes originally established by Baier; (2) the modal characterisation of bisimilarity by Desharnais et al. admits a much simpler completeness proof. Furthermore, inspired by the work of Hermanns et al. on probabilistic systems, we provide a sound and complete modal characterisation of fuzzy bisimilarity.

1 Introduction

The analysis of fuzzy systems has been the subject of active research during the last sixty years, and many formalisms have been proposed to model them: fuzzy automata (see, for example, [3, 4, 8, 24, 26, 30, 37]), fuzzy Petri nets [31, 33], fuzzy Markov processes [5] and fuzzy discrete event systems [25, 32].

Recently, a new formal model for fuzzy systems, fuzzy labelled transition systems (fLTSs, for short), has been proposed [15, 7, 21]. fLTSs are a natual generalization of the classical labelled transition systems in computer science in that after performing some action a system evolves from one state to a fuzzy set of successor states instead of a unique state. Many formal description tools for fuzzy systems, such as fuzzy Petri nets and fuzzy discrete event systems [25, 32], are not fLTSs. However, it is possible to translate a system's description in one of these formalisms into a fLTS to represent its behaviour.

Bisimulation [28] has been investigated in depth in process algebras because it offers a convenient co-inductive proof technique to establish behavioural equivalence [27]. It has been mostly used for verifying formal systems and is the foundation of state-aggregation algorithms that compress models by merging bisimilar states. State aggregation is routinely used as a preprocessing step before model checking [1, 17].

Errico and Loreti [15] proposed a notion of fuzzy bisimulation and applied it to fuzzy reasoning; Kupferman and Lusting [22] defined a latticed simulation between two lattice-valued Kripke structures, and applied it to latticed games; Cao et al. [7] defined a fuzzy bisimulation relation between two different fLTSs by a correlational pair based on some relation; Ćirić et al. [9] introduced four types of bisimulations (forward, backward, forward-backward, and backward-forward) for fuzzy automata.

All these approaches can be divided into two classes. In the first class, bisimulations are based on a crisp relation on the state space, so one state is either bisimilar to another state or not. As in [7, 15], the current paper falls into this class. In the second class, simulations or bisimulations are based on a fuzzy relation (or a lattice-valued relation) on the state space, which shows the degree of one state being bisimilar to another. This approach was adapted in [9, 22]. In addition, in [15] a bisimulation is necessarily an equivalence relation, which is not the case for [7] and the current work.

Following the seminal paper on exploring the connection between bisimulation and modal logic [19], a great amount of work has appeared that characterizes various kinds of bisimulations by appropriate logics. A logical characterizations of fuzzy bisimulation appeared in [15], which uses recursive formulas, and interprets a formula as a fuzzy set that gives the measure, respectively, of satisfaction and unsatisfaction of the formula.

In this paper we seek a variant of the Hennessy-Milner Logic (HML) to characterise fuzzy bisimulation. Since fuzzy systems are close to probabilistic systems, we revisit probabilistic bisimulaitons and their logical characterisations e.g. [13, 14, 20, 23]. We find that two results in probabilistic concurrency theory can be generalised or simplified.

- For finite-state reactive probabilistic labelled transition systems (rpLTSs) it is known that bisimilarity coincides with simulation equivalence. The result was originally proved by Baier [2], using techniques from domain theory. That proof is sophisticated and later on simplified by Zhang [39]. In the current work we generalise the above coincidence result to rpLTSs with *countable* state space. Our proof is surprisingly elementary and employs only some basic concepts of set theory.
- Desharnais et al. [14] proposed a probabilistic version of the HML to capture bisimilarity on rpLTSs. Their logic has neither negation nor infinite conjunctions, but is expressive enough to work for general rpLTSs that may have continuous state spaces. Their proof of this fact uses the machinery of analytic spaces. For rpLTSs with countable state spaces, we find that the completeness proof of their modal characterisation can be greatly simplified.

By adding negation and infinite conjunctions to the logic of Desharnais et al. we obtain a fuzzy variant of the HML to characterise bisimulation on general fLTSs. For fLTSs with countable state spaces and finite-support possibility distributions, infinite conjunctions can be replaced by binary conjunctions. The completeness proofs of our logical characterisations are inspired by Hermanns et al. [20]. Since the differences between the two models, fLTSs and rpLTSs, seem to be small. One is tempted to think that the current work might be a straightforward generalization of the above mentioned works for rpLTSs. However, this is not the case. For rpLTSs, negation is unnecessary to characterize bisimulation and binary conjunction is already sufficient, whereas for fLTSs both negation and infinite conjunction are necessary to characterize bisimulation for general fLTSs that may be infinitely branching. Therefore, fLTSs resemble more to nondeterministic systems rather than to deterministic systems in this aspect. Moreover, different techniques are needed for proving characterization theorems for fLTSs and rpLTSs. For example, in the case of rpLTSs the famous - theorem [6] holds, which greatly simplifies the completeness proof of the logical characterization. However, the - theorem is invalid for fLTSs, so we adopt a different approach to proving completeness: the basic idea is to construct a characteristic formula for each equivalence class, i.e. the formula is satisfied only by the states in that equivalence class. See Section 6 for more details.

The rest of this paper is structured as follows. We briefly review some basic concepts used in this paper in Section 2. Section 3 is devoted to showing the coincidence of bisimilarity with simulation equivalence in countable rpLTSs. Section 4 gives a modal characterization of probabilistic bisimulation for rpLTSs. Fuzzy bisimulation is introduced in Section 5 and a modal characterisation is provided in Section 6. Finally, this paper is concluded in Section 7.

2 Preliminaries

Let S be a set and $\mu : S \rightarrow [0, 1]$ a fuzzy set. The *support* of μ is the set $supp(\mu) = \{s \in S \mid \mu(s) > 0\}$. We denote by $\mathcal{F}(S)$ the set of all fuzzy sets in S, and $\mathcal{F}_f(S)$ the set of all fuzzy sets with finite supports, i.e. $\mathcal{F}_f(S) = \{\mu \in \mathcal{F}(S) \mid supp(\mu)$ is finite$\}$. With a slight abuse of notations, we sometimes write a possibility distribution to mean a fuzzy set[3].

A probability distribution on S is a fuzzy set Δ with $\sum_{s \in S} \Delta(s) = 1$. We write $\mathcal{D}(S)$ for the set of all probability distributions on S. We use $\overline{s}$ to denote the point distribution, satisfying $\overline{s}(t) = 1$ if $t = s$, and 0 otherwise. If $p_i \geq 0$ and Δ_i is a distribution for each i in some index set I, then $\sum_{i \in I} p_i \cdot \Delta_i$ is given by

$$(\sum_{i \in I} p_i \cdot \Delta_i)(s) = \sum_{i \in I} p_i \cdot \Delta_i(s)$$

If $\sum_{i \in I} p_i = 1$ then this is easily seen to be a distribution in $\mathcal{D}(S)$.

Let S be a set. For a binary relation $\mathcal{R} \subseteq S \times S$ we write $s \, \mathcal{R} \, t$ if $(s, t) \in \mathcal{R}$. A *preorder* relation $\mathcal{R}$ is a reflexive and transitive relation; an *equivalence* relation is a reflexive, symmetric and transitive relation. An equivalence relation $\mathcal{R}$ partitions a set S into equivalence classes. For $s \in S$ we use $[s]_{\mathcal{R}}$ to denote the unique equivalence class containing s. We drop the subscript $\mathcal{R}$ if the relation considered is clear from the context. Let $\mathcal{R}(s)$ denote the set $\{s' \mid (s, s') \in \mathcal{R}\}$. A set U is said to be $\mathcal{R}$-closed if $\mathcal{R}(s) \subseteq U$ for all $s \in U$, i.e. the image of U under $\mathcal{R}$, written $\mathcal{R}(U)$, is contained in U. We let $\mathcal{R}^*$ be the reflexive transitive closure of $\mathcal{R}$. Note that if $\mathcal{R}$ is a preorder then $\mathcal{R}^*$ coincides with $\mathcal{R}$. For any $s \in S$, the set $\mathcal{R}^*(s)$ is clearly a $\mathcal{R}$-closed set.

Definition 1. *A* fuzzy labelled transition system (fLTS) *is a triple* $(S, A, \rightarrow)$ *where* S *is a countable set of states,* A *is a set of actions, and the transition relation* $\rightarrow$ *is a partial function from* $S \times A$ *to* $\mathcal{F}(S)$*. If the transition relation* $\rightarrow$ *is a partial function from* $S \times A$ *to* $\mathcal{D}(S)$*, we say the fLTS is a* reactive probabilistic labelled transition system (rpLTS).

We sometimes write $s \xrightarrow{a} \mu$ and $s \xrightarrow{a[\mu]} s'$ for $\rightarrow(s, a) = \mu$ and $\rightarrow(s, a)(s') = \mu$, respectively. An fLTS $(S, A, \rightarrow)$ is said to be *image-finite* if for each state s and label a, we have $\rightarrow(s, a) \in \mathcal{F}_f(S)$.

[3] Strictly speaking a possibility distribution is different from a fuzzy set, though the former can be viewed as the generalized characteristic function of the latter. See [38] for more detailed discussion.

The fLTSs defined above are deterministic in the sense that for each state s and label a, there is at most one possibility distribution Δ with $s \xrightarrow{a} \Delta$. The rpLTSs defined above are usually called reactive probabilistic systems [18] or labelled Markov chains [13] in probabilistic concurrency theory. Similar to simple probabilistic automata [36], one could also define nondeterministic fuzzy transition systems by allowing for more than one transitions labelled with a same action leaving from a state.

3 Probabilistic Bisimulation and Simulation Equivalence

Let s and t are two states in a pLTS, we say t can simulate the behaviour of s if the latter can exhibit action a and lead to distribution Δ then the former can also perform a and lead to a distribution, say Θ, which can mimic Δ in successor states. We are interested in a relation between two states, but it is expressed by invoking a relation between two distributions. To formalise the mimicking of one distribution by the other, we make use of a lifting operation by following [11].

Definition 2. *Given a set S and a relation $R \subseteq S \times S$, we define $R^{\dagger} \subseteq D(S) \times D(S)$ as the smallest relation that satisfies:*

1. *$s \mathrel{R} t$ implies $\overline{s} \mathrel{R^{\dagger}} \overline{t}$*
2. *$\Delta_i \mathrel{R^{\dagger}} \Theta_i$ implies $(\sum_{i \in I} p_i \cdot \Delta_i) \mathrel{R^{\dagger}} (\sum_{i \in I} p_i \cdot \Theta_i)$, where I is an index set and $\sum_{i \in I} p_i = 1$.*

The proposition below is immediate.

Proposition 1. *Let Δ and Θ be two distributions over S, and $R \subseteq S \times S$. Then $\Delta \mathrel{R^{\dagger}} \Theta$ if and only if Δ, Θ can be decomposed as follows:*

1. *$\Delta = \sum_{i \in I} p_i \cdot \overline{s_i}$, where I is an index set and $\sum_{i \in I} p_i = 1$*
2. *For each $i \in I$ there is a state t_i such that $s_i \mathrel{R} t_i$*
3. *$\Theta = \sum_{i \in I} p_i \cdot \overline{t_i}$.*

An important point here is that in the decomposition of Δ into $\sum_{i \in I} p_i \cdot \overline{s_i}$, the states s_i are not necessarily distinct: that is, the decomposition is not in general unique. Thus when establishing the relationship between Δ and Θ a given state s in Δ may play a number of different roles.

If R is an equivalence relation, the lifted relation $R^{\dagger}$ can be defined alternatively as given in the orginal work by Larsen and Skou [23].

Proposition 2. *Let Δ, Θ be two distributions over S and R is an equivalence relation. Then $\Delta \mathrel{R^{\dagger}} \Theta$ if and only if $\Delta(C) = \Theta(C)$ for each equivalence class $C \in S/R$, where $\Delta(C)$ stands for the accumulation probability $\sum_{s \in C} \Delta(s)$.*

Proof. See Theorem 2.4 (2) in [10]. ⊓⊔

Lemma 1. *Let $\Delta, \Theta \in D(S)$ and R be a binary relation on S. If $\Delta \mathrel{R^{\dagger}} \Theta$ then we have $\Delta(A) \leq \Theta(R(A))$ for each set $A \subseteq S$.*

Proof. Since $R^{\dagger}$, by Proposition 1 we can decompose and as follows.

$$= \sum_{i \in I} p_i \overline{s_i} \quad \overline{s_i}\ R\ t_i \quad = \sum_{i \in I} p_i \overline{t_i}$$

Note that $\{s_i\}_{i \in I} = \lceil \ \rceil$. We define an index set J I as follows: $J = \{i \in I \mid s_i \in A\}$. Then $(A) = \{p_j\}_{j \in J}$. For each $j \in J$ we have $s_j\ R\ t_j$, i.e. $t_j \in R(s_j)$. It follows that $\{t_j\}_{j \in J}$ $R(A)$. Therefore, we can infer that

$$(A) = (\{s_j\}_{j \in J}) = \sum_{j \in J} p_j = (\{t_j\}_{j \in J}) \quad (R(A)).$$

Remark 1. The converse of Lemma 1 also holds [34], though it is not used in this paper. So an alternative way of lifting relations [12] is to say that is related to by lifting R if (A) $(R(A))$ for each set A S.

Corollary 1. *Let* $;$ $\in D(S)$ *and* R *be a binary relation on* S. *If* $R^{\dagger}$ *then* (A) (A) *for each* R*-closed set* A S.

Proof. Let A S be R-closed. Then we have $R(A)$ A, and thus $(R(A))$ (A). By Lemma 1, if $R^{\dagger}$ then (A) $(R(A))$. It follows that (A) (A). ∎

Lemma 2. *Let* R *be a preorder on a set* S *and* $;$ $\in D(S)$. *If* $R^{\dagger}$ *and* $R^{\dagger}$ *then* $(C) = (C)$ *for all equivalence classes* C *with respect to the kernel* $R \cap R^{-1}$ *of* R.

Proof. Let us write for $R \cap R^{-1}$. For any $s \in S$, let $[s]$ be the equivalence class that contains s. Let A_s be the set $\{t \in S \mid s\ R\ t \wedge t \not R\ s\}$. It holds that

$$\begin{aligned} R(s) &= \{t \in S \mid s\ R\ t\} \\ &= \{t \in S \mid s\ R\ t \wedge t\ R\ s\} \uplus \{t \in S \mid s\ R\ t \wedge t \not R\ s\} \\ &= [s] \uplus A_s \end{aligned}$$

where $\uplus$ stands for a disjoint union. Therefore, we have

$$(R(s)) = ([s]) + (A_s) \quad \text{and} \quad (R(s)) = ([s]) + (A_s) \tag{1}$$

We now check that both $R(s)$ and A_s are R-closed sets, that is $R(R(s))$ $R(s)$ and $R(A_s)$ A_s. Suppose $u \in R(R(s))$. Then there exists some $t \in R(s)$ such that $t\ R\ u$, which means that $s\ R\ t$ and $t\ R\ u$. As a preorder R is a transitive relation. So we have $s\ R\ u$ which implies $u \in R(s)$. Therefore we can conclude that $R(R(s))$ $R(s)$.

Suppose $u \in R(A_s)$. Then there exists some $t \in A_s$ such that $t\ R\ u$, which means that $s\ R\ t$, $t \not R\ s$ and $t\ R\ u$. As a preorder R is a transitive relation. So we have $s\ R\ u$. Note that we also have $u \not R\ s$. Otherwise we would have $u\ R\ s$, which means, together with $t\ R\ u$ and the transitivity of R, that $t\ R\ s$, a contradiction to the hypothesis $t \not R\ s$. It then follows that $u \in A_s$ and then we conclude that $R(A_s)$ A_s.

We have verified that $R(s)$ and A_s are R-closed sets. Since $R^{\dagger}$ and $R^{\dagger}$, we apply Corollary 1 and obtain that $(R(s))$ $(R(s))$ and $(R(s))$ $(R(s))$, that is

$$(R(s)) = (R(s)) \tag{2}$$

Similarly, using the fact that A_s is R-closed we obtain that

$$(A_s) = (A_s) \tag{3}$$

It follows from (1)-(3) that

$$([s]\ ) = ([s]\ )$$

as we have desired. ⊓⊔

A restricted version of the above lemma (by requiring the state set S to be finite) has appeared as Lemma 5.3.5 in [2], but the proof given there is much more complicated as it relies on some properties of DCPOs, which is then simplified in [39]. In this paper, we allow the state set of a rpLTS to be a countably *infinite* set. With this key technical lemma at hand, we are ready to prove the coincidence of simulation equivalence and bisimilarity, which was originally given as Theorem 5.3.6 in [2].

Definition 3. *A relation R S S is a* probabilistic simulation *if s R t and $s \xrightarrow{a}$ implies that some exists such that $t \xrightarrow{a}$ and $R^{\dagger}$. If both R and R^{-1} are probabilistic simulations, then R is a* probabilistic bisimulation*. The largest probabilistic bisimulation, denoted by $_p$, is called* probabilistic bisimilarity*. The largest probabilistic simulation, denoted by $_p$, is called* probabilistic similarity*. The kernel of probabilistic similarity, i.e $_p \cap\ _p^{-1}$, is called* simulation equivalence*, denoted by $_p$.*

In general, simulation equivalence is coarser than bisimilarity. However, for rpLTSs, the two relations do coincide.

Theorem 1. *For rpLTSs, simulation equivalence coincides with bisimilarity.*

Proof. It is obvious that $_p$ is included in $_p$. For the other direction, we show that $_p$ is a bisimulation. Let $s, t \in S$ and s t. Suppose that $s \xrightarrow{a}$. There exists a transition $t \xrightarrow{a}$ with $(\ _p)^{\dagger}$. Since we are considering reactive probabilistic systems, the transtion $t \xrightarrow{a}$ from t must be matched by the transiton $s \xrightarrow{a}$ from s, with $(\ _p)^{\dagger}$. Note that $_p$ is obviously a preorder on S. It follows from Lemma 2 that $(C) = (C)$ for any $C \in S/\ _p$. Since $_p$ is clearly an equivalence relation, by Proposition 2 we see that $(\ _p)^{\dagger}$. Therefore, $_p$ is indeed a bisimulation relation. ⊓⊔

4 Modal Characterisation of Probabilistic Bisimulation

Let A be a set of *actions* ranged over by $a, b, $, and let $\top$ be a propositional constant. The language L_p of formulas is the least set generated by the following BNF grammar:

$$\varphi ::= \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle_p \varphi .$$

where a is an action and p is a rational number in the unit interval $[0,1]$. This is the basic logic with which we establish the logical characterization of bisimulation.

Let us fix a rpLTS $(S, A, \rightarrow)$. The semantic interpretation of formulas in $\mathcal{L}_p$ is given by:

- $s \models_p \top$, for any state s;
- $s \models_p \varphi_1 \wedge \varphi_2$, if $s \models_p \varphi_1$ and $s \models_p \varphi_2$;
- $s \models_p \langle a \rangle_p \varphi$, if $s \xrightarrow{a} \mu$ and $\exists A \subseteq S.\ (\forall s' \in A.\ s' \models_p \varphi) \wedge (\mu(A) \geq p).$

We write $[\![\varphi]\!]$ for the set $\{s \in S \mid s \models_p \varphi\}$. Then it is immediate that $s \models_p \langle a \rangle_p \varphi$ iff $s \xrightarrow{a} \mu$ and $\mu([\![\varphi]\!]) \geq p$, i.e. $\sum_{s' \in [\![\varphi]\!]} \mu(s') \geq p$. Thus $s \models_p \langle a \rangle_p \varphi$ says that the state s can make an a-move to a distribution that evolves into a state satisfying φ with probability at least p. In the sequel we always use this fact as the semantic interpretation of the formula $\langle a \rangle_p \varphi$ in $\mathcal{L}_p$.

The logic above induces a logical equivalence relation between states.

Definition 4. *Let s and t be two states in a rpLTS. We write $s =_p t$ if $s \models_p \varphi \Leftrightarrow t \models_p \varphi$ for all $\varphi \in \mathcal{L}_p$.*

The following lemma says that the transition probabilities to sets of the form $[\![\psi]\!]$ are completely determined by the formulas. It has appeared as Lemma 7.7.6 in [35].

Lemma 3. *Let s and t be two states in a rpLTS. If $s =_p t$ and $s \xrightarrow{a} \mu$, then some ν exists with $t \xrightarrow{a} \nu$, and for any formula $\psi \in \mathcal{L}_p$ we have $\mu([\![\psi]\!]) = \nu([\![\psi]\!])$.*

Proof. First of all, the existence of ν is obvious because otherwise the formula $\langle a \rangle_1 \top$ would be satisfied by s but not by t.

Let us assume, without loss of generality, that there exists a formula ψ such that $\mu([\![\psi]\!]) < \nu([\![\psi]\!])$. Then we can squeeze in a rational number p with $\mu([\![\psi]\!]) < p \leq \nu([\![\psi]\!])$. It follows that $t \models_p \langle a \rangle_p \psi$ but $s \not\models_p \langle a \rangle_p \psi$, which contradicts the hypothesis that $s =_p t$. ⊓⊔

We will show that the logic $\mathcal{L}_p$ can characterise bisimulation. The completeness proof of the characterisation crucially relies on the π-λ theorem [6]. Let P be a family of subsets of a set X. We say P is a π-class if it is closed under finite intersections; P is a λ-class if it is closed under complementations and countable disjoint unions.

Theorem 2 (The π-λ theorem). *If P is a π-class, then $\sigma(P)$ is the smallest λ-class containing P, where $\sigma(P)$ is a σ-algebra containing P.*

The next proposition is a typical application of the π-λ theorem [16], which tells us that when two probability distributions agree on a π-class they also agree on the generated σ-algebra.

Proposition 3. *Let S be a state space, $\mathcal{A}_0 = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}_p\}$, and $\mathcal{A} = \sigma(\mathcal{A}_0)$. For any $\mu, \nu \in D(S)$, if $\mu(A) = \nu(A)$ for any $A \in \mathcal{A}_0$, then $\mu(B) = \nu(B)$ for any $B \in \mathcal{A}$.*

Proof. Let $P = \{A \in \mathcal{A} \mid \mu(A) = \nu(A)\}.$ Then P is closed under countable disjoint unions because probability distributions are σ-additive. Furthermore, $\mu(S) = \nu(S) = 1$ implies that if $A \in P$ then $\mu(S \setminus A) = \mu(S) - \mu(A) = \nu(S) - \nu(A) = \nu(S \setminus A)$, i.e. $S \setminus A \in P$. Thus P is closed under complementation as well. It follows that P is a λ-class. Note that $\mathcal{A}_0$ is a π-class in view of the equation $\llbracket \varphi_1 \wedge \varphi_2 \rrbracket = \llbracket \varphi_1 \rrbracket \cap \llbracket \varphi_2 \rrbracket$. Since $\mathcal{A}_0 \subseteq P$, we can apply the π-λ Theorem to obtain that $\mathcal{A} = \sigma(\mathcal{A}_0) \subseteq P \subseteq \mathcal{A}$, i.e. $\mathcal{A} = P$. Therefore, $\mu(B) = \nu(B)$ for any $B \in \mathcal{A}$. □

Theorem 3. *Let s and t be two states in a rpLTS. Then $s \sim_p t$ iff $s =_p t$.*

Proof. The proof of soundness is carried out by a routine induction on the structure of formulas. Below we focus on the completeness. It suffices to show that $=_p$ is a bisimulation. Note that $=_p$ is clearly an equivalence relation. For any $u \in S$ the equivalence class in $S/_{=_p}$ that contains u is

$$[u] = \bigcap \{\llbracket \varphi \rrbracket \mid u \models_p \varphi\} \cap \bigcap \{S \setminus \llbracket \varphi \rrbracket \mid u \not\models_p \varphi\}. \qquad (4)$$

In (4) only countable intersections are used because the set of all the formulas in the logic $\mathcal{L}_p$ is countable. Let $\mathcal{A}_0$ be defined as in Proposition 3. Then each equivalence class of $S/_{=_p}$ is a member of $\sigma(\mathcal{A}_0)$.

On the other hand, $s =_p t$ and $s \xrightarrow{a} \mu$ implies that some distribution ν exists with $t \xrightarrow{a} \nu$ and for any $\varphi \in \mathcal{L}_p$, $\mu(\llbracket \varphi$